Representing Simplicial Complexes with Mangroves

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The Mangrove Topological Data Structure Framework

What we propose - The Mangrove TDS Framework

- ***Fast prototyping*** of topological data structures for simplicial complexes under the same API.
- ***Extensible*** and graph-based representations of connectivity information (***mangroves***).
- ***Implicit representations*** of those simplices, not directly encoded (***ghost simplices***).

We have designed ***several*** data structures with ***complementary properties*** including:

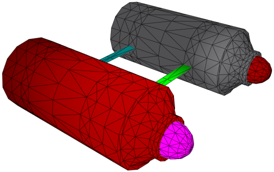
- the *Incidence Simplicial (IS)* data structure, *De Floriani et all., IMR 2010*
- the IA^* data structure, *Canino et all., SMI 2011*

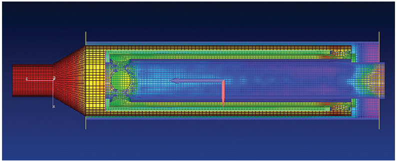
which are ***efficient*** for manipulating ***non-manifolds*** of any dimension (with pieces of ***different dimensionality***), *Canino, 2012* (PhD. Thesis)

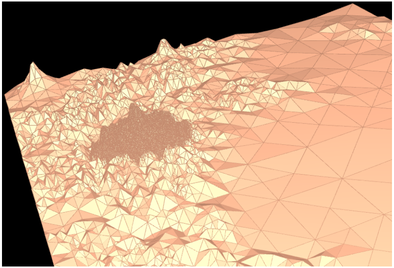
Our C++ implementations are contained in the ***Mangrove TDS Library***, released as GPL v3 software for the scientific community at

`http://mangrovetds.sourceforge.net`

Context & Motivation







Starting point

- ***Simplicial complexes*** are common in many applications (FEM, C&G, . . .)
- ***Efficient representations*** for extracting and modifying their ***connectivity information***

What we have?

Many ***topological data structures***:

- *Garimella, 2002 + course*
- *De Floriani and Hui, 2005 (survey)*
- *De Floriani et all., 2010*
- *Canino et all., 2011*
- More references in our paper!

optimized for a ***specific task***.

Main Issue

Not much attention towards ***rapid prototyping*** of topological data structures under the ***same API***.

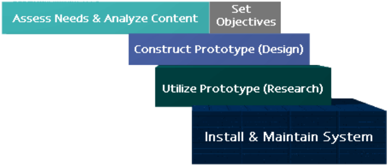
Context & Motivation (con’td)

Very few ***frameworks*** support ***rapid prototyping*** of representations:

	OpenMesh	OpenVolumeMesh	CGoGN	CGAL	Mangrove TDS
Types of shapes	cell	cell	cell	cell	simplicial/cell
Dimension of shapes	2	up to 3	any	any	any
Representation	OM	OVM	C-Maps	many	any
Extensibility	no	no	no	modules	yes
Non-manifolds	partially	yes	partially	yes	yes

Rapid prototyping is important in the manifacturing.

Rapid Prototyping
Design Model



From "A Conceptual Framework for Comparing Instructional Design Models"

What do we want in this context?

Generic prototype of topological data structures:

- ***customized*** in order to simulate any topological representation efficiently (***design***)
- ***dynamically replaced*** at run-time (plugin) wrt any modeling need (***optimization***)
- well-defined and unique API (***data hiding***)
- ***short learning curve*** and ***easy-to-use***.

Context & Motivation (con'td)

Advantages?

- ***Complete analysis*** of topological data structures within the same context
- ***Common basis*** for coherent and fair ***comparisons*** wrt efficiency of topological queries
- ***Simplification*** of programmers' work (unique API)
- ***No rewriting*** a program from scratch when replacing internal representations completely.
- Possibility to ***replace completely*** representations, not only few details.
- Simulations of representations without a ***relevant overhead***.

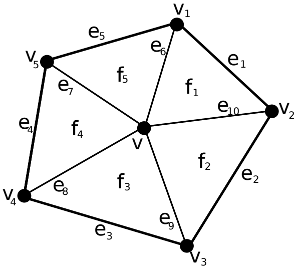
This can be defined by exploiting ***topological relations*** for any simplicial (cell) complex:

- any domain, including ***non-manifolds***
- pieces of ***different dimensionality***
- any dimension (***dimension-independent***)

Representing the Connectivity Information

Key idea

Connectivity of simplices (vertices, edges, triangles,. . .) in any simplicial complex Σ (1-skeleton, tri, and tet meshes,. . .) is described by ***topological relations***.



Topological relation $\sim_{k,m}(\sigma, \sigma')$

Let Σ^j be the collection of j-simplices in Σ, and $\sim_{k,m} \subseteq \Sigma^k \times \Sigma^m$, then:

Boundary relations

If σ' is a face of σ, e.g., $\sim_{2,0}(f_1, v)$ or $\sim_{2,1}(f_3, e_8)$

Co-boundary relations (incident simplices)

If σ is a face of σ', e.g., $\sim_{0,2}(v, f_1)$ or $\sim_{1,2}(e_8, f_3)$

Adjacency relations

- $\sim_{k,k}(\sigma, \sigma')$, with $k \neq 0$, if σ and σ' shares a $(k-1)$-simplex
- $\sim_{0,0}(\sigma, \sigma')$, if an edge connects σ and σ'

For instance, $\sim_{0,0}(v, v_3)$ or $\sim_{2,2}(f_1, f_3)$

Representing the Connectivity Information as a Graph

Key Idea

Connectivity information can be described as a ***directed graph*** $\mathcal{G}_\Sigma = \{\mathcal{N}_\Sigma, \mathcal{A}_\Sigma\}$ where:

- each node n_σ corresponds to a simplex σ
- each oriented arc $(n_\sigma, n_{\sigma'})$ corresponds to $\sim_{k,m}(\sigma, \sigma')$

Graph $\mathcal{G}_\Sigma$ is ***expensive*** to be encoded, but it is our ***generic prototype***!

Topological Data Structure $\mathcal{M}_\Sigma$

- Subset of ***simplices***
- Subset of ***topological relations***:
  - *boundary* relations
  - *co-boundary* relations
  - *adjacency* relations

$\rightarrow$

Mangrove $\mathcal{G}_\Sigma^{\mathcal{M}}$

- Graph-based representation of $\mathcal{M}_\Sigma$ (as above)
- ***Spanning subgraph*** of $\mathcal{G}_\Sigma$, formed by:
  - *boundary* graph
  - *co-boundary* graph
  - *adjacency* graph

Consequence

Representing any ***topological data structure*** $\mathcal{M}_\Sigma \equiv$ representing ***mangrove*** $\mathcal{G}_\Sigma^{\mathcal{M}}$, regardless:

- what simplices and topological relations are encoded in $\mathcal{M}_\Sigma$
- the dimension and the type of domain

Everything is a mangrove!

Generic Encoding of Mangroves and Operations

Any ***mangrove*** $\mathcal{G}_\Sigma^\mathcal{M}$ is encoded by any ***adjacency list*** data structure for graphs (well known).

Encoding of Node $n_\sigma \equiv$ simplex σ

- ***Unique*** topological entity to be encoded
- Endpoints of ***arcs*** in $\mathcal{G}_\Sigma^\mathcal{M}$ outgoing from n_σ
- ***Properties*** associated with nodes

Collection of Nodes (indices-based)

- ***Dynamic arrays***, one for each collection of simplices in $\mathcal{M}_\Sigma$
- ***Garbage collector*** and ***safe iterators***

This encoding satisfies requirements in *Sieger & Botsch, IMR 2011*.

Common API (at least)

Topological queries on any simplex σ:

- BOUNDARY - faces of σ
- STAR - simplices incident at σ
- ADJACENCY - simplices adjacent to σ
- LINK - boundary of STAR(σ), not incident at σ
- IS-MANIFOLD - checking if σ is manifold

are ***breadth-first traversals*** of $\mathcal{G}_\Sigma^\mathcal{M}$.

What we need for modeling $\mathcal{M}_\Sigma$?

- Only the content of ***mangrove*** $\mathcal{G}_\Sigma^\mathcal{M}$
- Implementation of ***topological queries***

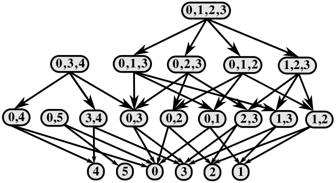
Consequences (except some cases)

- No ***overhead*** for simulating $\mathcal{M}_\Sigma$
- Transparent API (data ***hiding***)
- ***Plugin***

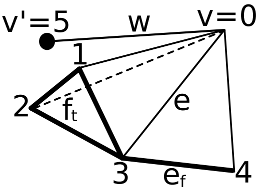
The IS data structure (De Floriani et all., IMR 2010)

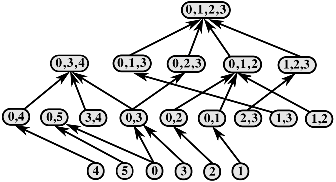
The IS data structure (The IS-graph)

- Simplicial complexes with pieces of ***different dimensionality***
- Dimension-independent
- ***All*** simplices are encoded
- ***Global*** mangrove



Immediate Boundary relations $\sim_{p,p-1}$ for any p-simplex



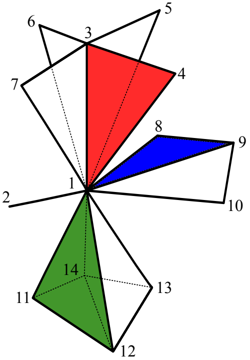


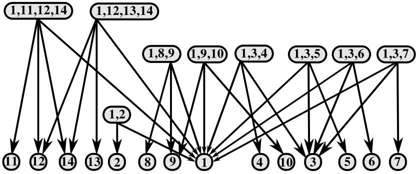
Co-boundary relations $\sim^*_{p,p+1}$ for any p-simplex

The IA* data structure (Canino et all., SMI 2011)

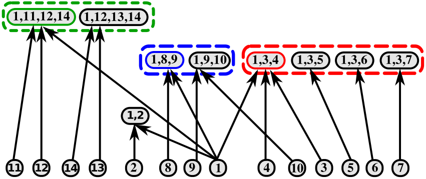
The IA* data structure (The IA*-graph)

- Extends the ***Indexed data structure*** to non-manifolds of any dimension
- Simplicial complexes with pieces of ***different dimensionality***
- Only ***vertices*** and ***top simplices*** (no simplices incident at)
- ***Partial*** mangrove

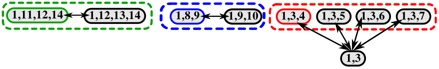




Boundary relations $\sim_{p,0}$ for top simplices



Co-boundary relations $\sim^*_{0,p}$ for vertices



Adjacency relations $\sim^*_{p,p}$ and $\sim^*_{p-1,p}$ for top simplices

Implicit Representations of Simplices

API Problem with the IA* data structure

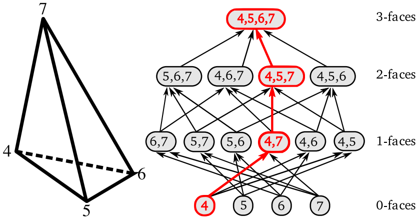
- ***Not all*** simplices are encoded in $\mathcal{M}_\Sigma$
- What's their representation in the API?

Solution

- ***Implicit*** representation of these simplices
- ***Simple*** and ***efficient*** in space and time

First attempt: Cell Tuples

- Based on the ***incidence graph (IG)***, *Edelsbrunner, 1987* for d-complexes
- ***Maximal paths*** from vertices to top simplices in the IG
- ***Redundant*** connectivity information
- Length proportional to d, ***no scalable***



Cell Tuple (in red)

Solving Drawbacks . . .

- ***Adjacency relations*** are restricted to ***top simplices*** (recall)
- ***Quick connections*** among a simplex σ and top simplices incident at σ
- Avoiding ***complete traversals*** in the incidence graph

Ghost Simplices

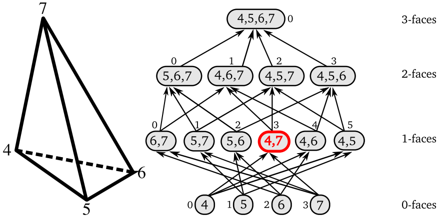
Key Idea

- Any p-simplex σ is a ***p-face*** of an arbitrary ***top t-simplex*** σ' incident at σ
- ***Enumeration*** of simplices and faces of σ'

Definition

Always defined as a 4-tuple $[t, i, p, j]$, where:

- t is the dimension of σ'
- i is the unique identifier of σ'
- p is the dimension of σ
- j is the local identifier of σ as p-face of σ'



Ghost simplex [3, 0, 1, 3] (in red)
Ghost simplex [3, 0, 3, 0] is the top 3-simplex

Properties

- ***scalable*** to high dimensions
- ***one*** ghost simplex for each ***top simplex*** incident at σ (flights in sharing code)

Consequences

- ***Input*** and ***output*** of topological queries are ***ghost simplices***
- New ***organization*** for topological queries

Ghost Simplices (cont'd)

New Organization of Topological Queries

1. retrieve all ***top simplices*** σ' incident at σ
2. retrieve ***faces*** of σ' involved in the query of interest (as ***ghost simplices*** wrt σ')

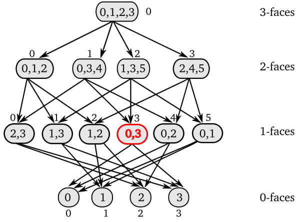
What we encode

For each dimension t of top simplices:

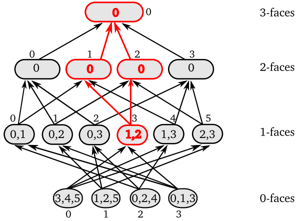
- ***incidence graph*** $\mathcal{H}_t$ describing topology of ***faces*** (ghost simplices) for a top t-simplex
- local indices j of faces stored in $\mathcal{H}_t$

Consequences

Faces of interest are retrieved by ***breadth-first traversals*** of $\mathcal{H}_t$



Boundary relations

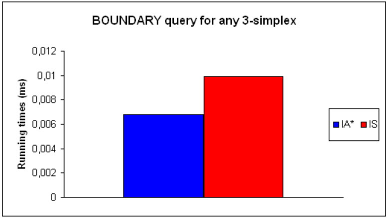


Co-boundary relations

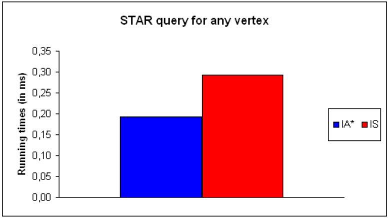
Faces [3, x, 2, 1], [3, x, 2, 2], and [3, x, 3, 0] are incident at [3, x, 1, 3].

Using Ghost Simplices in our Mangrove TDS Library

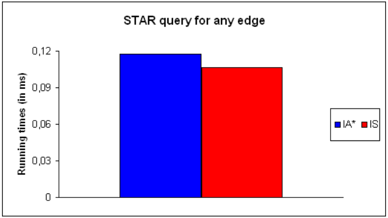
Ghost simplices improves the ***efficiency*** of topological queries in the IA^* data structure:



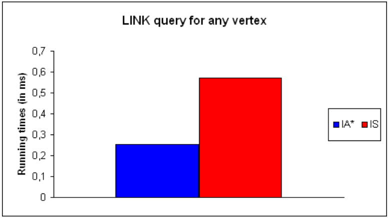
30% faster



30% faster



only 10% slower



up to 3*X* faster

The ***Mangrove TDS Library*** is a GPL v3 software: `http://mangrovetds.sourceforge.net`

Using Ghost Simplices . . . (cont'd)

Overhead $\mathcal{C}^d$ for graphs $\mathcal{H}_t$

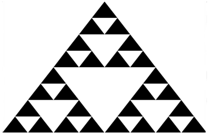
$$\sum_{t=2}^{p}\sum_{p=2}^{t}(p+1)\binom{t+1}{p+1}$$

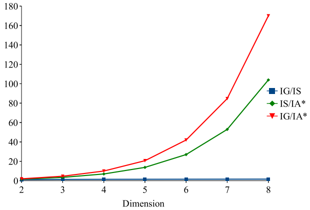
- does not depend on the number of top simplices
- much less than 1%

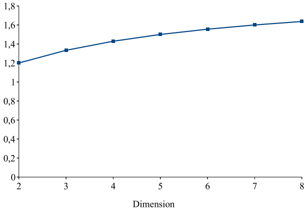
Storage costs of the IG, IS, and IA* data structures within our framework:

d	$\mathcal{C}^d$	s_0	s_d	$\mathcal{S}_{IA*}$	$\mathcal{S}_{IS}$	$\mathcal{S}_{IG}$
2	18	$2.8M$	$5.6M$	$22.4M$	$38M$	$44.8M$
3	74	$1.4M$	$4.2M$	$19.6M$	$68.6M$	$92.1M$
4	224	$0.7M$	$2.7M$	$14.9M$	$104.3M$	$149M$
5	596	$0.3M$	$1.4M$	$9M$	$123.6M$	$188M$
6	$1.5k$	$0.1M$	$0.7M$	$5.3M$	$143.1M$	$222.6M$
7	$3.5k$	$75K$	$0.5M$	$4.3M$	$228M$	$365.5M$
8	$8.1k$	$34K$	$0.3M$	$2.5M$	$260M$	$425M$

Generalization of the *Sierpinski* shape in high dimensions







Small ***overhead*** implies great ***expressive power*** and ***compactness*** in high dimensions.

Conclusions & Future Work

What we proposed - The Mangrove TDS Framework

- ***Fast prototyping*** of topological data structures for simplicial complexes under the same API.
- ***Extensible*** and graph-based representations of connectivity information (***mangroves***).
- ***Implicit representations*** of those simplices, not directly encoded (***ghost simplices***).

The ***Mangrove TDS Library***, released as GPL v3 software for the scientific community at

`http://mangrovetds.sourceforge.net`

There is a lot of space and room for ***improvements***:

- new ***plugins***
- extensions to ***cell complexes*** (e.g., quad and hex meshes)
- new ***editing operators*** (homology-preserving and -modifying operators)
- applications in ***high dimensions***
- ***distributed*** and ***parallel*** version
- . . .

Acknowledgements

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- Prof. Vijay Natarajan, Indian Institute of Science, for the *Sierpinski* shape

These slides are available on `http://www.disi.unige.it/person/CaninoD`

Thank you much for your attention!