A Compact Representation for Topological Decompositions of Non-Manifold Shapes

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Abstract: Simplicial complexes are extensively used for discretizing digital shapes in several applications. A structural description of a non-manifold shape can be obtained by decomposing the input shape into a collection of meaningful components with a simpler topology. Here, we consider a unique and dimension-independent decomposition of a non-manifold shape into nearly manifold components, known as the *Manifold-Connected (MC-) decomposition*. We present the *Compact Manifold-Connected (MC-) graph*, an efficient graph-based representation for the MC-decomposition, which can be combined with any topological data structure for encoding the underlying components. We present the main properties of this representation as well as algorithms for its generation. We also show that this representation is more compact than several topological data structures, which do not explicitly describe the non-manifold structure of a shape.

1 INTRODUCTION

Modeling digital shapes requires efficient representations, which integrate geometric, topological, and semantic aspects. A digital shape is often discretized by any simplicial complex. A very common representation of this latter is provided by *topological data structures*, which encode a subset of simplices, and the connectivity information among such simplices. In the literature, there is a large amount of research about these representations (De Floriani and Hui, 2005). In any case, they describe only combinatorial information of shapes, and do not expose their relevant components and their connectivity.

In order to overcome these limitations, a shape may be decomposed into *meaningful* components, which are easily distinguished from the remaining part of the object so as to reduce the complexity of a shape. The resulting representation highlights components of a shape, and their connectivity, namely it is a *structural model*. Here, geometric details are abstracted, and this model provides a high-level description of a shape. Hence, this model is a suitable basis for semantic annotation and reasoning.

Most structural models are defined for manifold shapes (Shamir, 2008). Informally, a *manifold* is a subset of the Euclidean space such that the neighborhood of each point is homeomorphic to an open ball. Non-manifold shapes do not satisfy this property at one or more points, which are called the *non-manifold singularities*. Non-manifold shapes arise in several applications, for instance, as the result of the *idealization* process during numerical simulations (Thakur et al., 2009). There is also an increasing interest for modeling non-manifold shapes, discretized through abstract simplicial complexes, which are not necessarily embedded in any Euclidean space, like the *Vietoris-Rips complexes* (Gromov, 1987).

The most natural decomposition of any non-manifold shape consists of cutting this latter along its non-manifold singularities without modifying the manifold parts. The resulting representation highlights manifold components and their connectivity, and non-manifold singularities are exposed explicitly. However, this decomposition is possible only for 2-complexes. In three or higher dimensions, it may introduce artificial "cuts" in the input shape, and create almost manifold components (De Floriani et al., 2003). In six or higher dimensions, this decomposition is not feasible, since the class of manifolds is not decidable (Nabutovsky, 1996).

Here, we describe a dimension-independent extension of the *Manifold-Connected (MC-) decomposition*, introduced in (Hui and De Floriani, 2007) for 2D and 3D shapes. This latter is a structural model for abstract simplicial complexes, which are decomposed as a collection of *Manifold-Connected (MC-) components*, which form a decidable superclass of manifolds. These components may contain non-manifold singularities, and thus they cannot be represented by topological data structures specific for manifolds.

In this paper, we propose a two-level graph-based representation for the MC-decomposition, which we call the *Compact Manifold-Connected (MC-) graph*. Here, the lower level consists of a topological data structure, capable to represent non-manifold shapes. In our experiments we have used the *Incidence Simplicial (IS)* data structure (De Floriani et al., 2010), and the *Generalized Indexed data structure with Adjacencies (IA*)* (Canino et al., 2011). The upper level consists of MC-components, expressed through references to simplices in the underlying data structure, and acts in the same way as a spatial index (Samet, 2006). Then, we present an algorithm for extracting the compact MC-graph from any topological data structure. We also show that the compact MC-graph, if combined with the IS and the IA* data structures, is more compact than a widely used data structure, namely the *Incidence Graph (IG)* (Edelsbrunner, 1987), which does not expose the structure of a shape explicitly. Finally, we also show that the compact MC-graph, combined with the IA* data structure, is more cost effective than the *Initial Quasi-Manifold (IQM-) decomposition* (De Floriani et al., 2003), and than the *Double Level Decomposition (DLD)* data structure (Hui et al., 2006), which contains the IQM-decomposition of simplicial 3-complexes, embedded in the Euclidean space $\mathbb{E}^3$.

The remainder of this paper is organized as follows. In Section 2, we present some background notions, while, in Section 3, we briefly discuss related work. In Section 4, we define the MC-decomposition for a non-manifold shape in arbitrary dimensions. In Section 5, we introduce and analyze our compact MC-graph, while, in Section 6, we present some experimental results. Finally, in Section 7, we draw concluding remarks and discuss future work.

2 BACKGROUND NOTIONS

In this section, we introduce background notions, which we will use throughout the paper.

A Euclidean *p-simplex* σ is the convex hull of a set V_σ of $p+1$ affine points in the Euclidean space $\mathbb{E}^n$, with $p \leq n$, and $p = dim(\sigma)$ is the *dimension* of σ. A k-simplex σ', with $k \leq p$, generated by $k+1$ vertices in V_σ, is a k-face of σ. Here, σ and σ' are *mutually incident*. A finite collection Σ of Euclidean simplices is a *Euclidean simplicial complex* if (i) all the faces of each simplex are in Σ, and (ii) for each pair of simplices σ and σ' in Σ, either $\sigma \cap \sigma' = \emptyset$, or $\sigma \cap \sigma'$ is in Σ. The maximum dimension d of the simplices in Σ is the *dimension* of Σ, which will thus be called a *simplicial d-complex*. The *domain* of a simplicial d-complex Σ embedded in $\mathbb{E}^n$, with $d \leq n$, is the subset of $\mathbb{E}^n$ spanned by all the simplices of Σ. A Euclidean simplicial complex Σ is the geometric realization of an *abstract simplicial complex*, which represents the combinatorial structure of Σ.

The *combinatorial boundary* $b(\sigma)$ of a simplex σ consists of all the faces of σ. The *star* $St(\sigma)$ is formed by all the simplices incident at σ. If $St(\sigma) = \emptyset$, then σ is a *top* simplex. Top d-simplices are called *maximal*. A simplicial d-complex Σ is *regular* if all the top simplices are maximal. The *link* $Lk(\sigma)$ of a simplex σ consists of all the faces of the simplices in $St(\sigma)$, which are not incident at σ. Two k-simplices are *adjacent* if they share a $(k-1)$-face, while two vertices are *adjacent* if they are connected by a common edge.

An *h-path* is any sequence of $(h+1)$-simplices $(\sigma_i)_{i=0}^{k}$ such that two consecutive simplices σ_{i-1} and σ_i are adjacent. Two simplices σ' and σ'' are *h-connected* if an h-path $(\sigma_i)_{i=0}^{k}$ connects σ' and σ''. A subcomplex Σ' of any simplicial complex Σ is *h-connected* if any two simplices in Σ' are h-connected.

A $(d-1)$-simplex σ in a regular simplicial d-complex is *combinatorial manifold*, if at most two d-simplices are incident at σ. A k-simplex σ, such that $Lk(\sigma)$ is homeomorphic to the $(d-k)$-sphere, is *combinatorial manifold*, otherwise σ is a *non-manifold singularity*. A simplicial d-complex, such that all vertices are combinatorial manifold, is a *combinatorial d-manifold* (De Floriani et al., 2003). Note that there are some algorithmically non-recognizable combinatorial d-manifolds for $d \geq 6$ (Nabutovsky, 1996), thus the class of combinatorial d-manifolds is not always decidable. A regular $(d-1)$-connected simplicial d-complex Σ, such that the star of each $(d-1)$-simplex consists of at most two d-simplices, is a *combinatorial pseudo-manifold*.

3 RELATED WORK

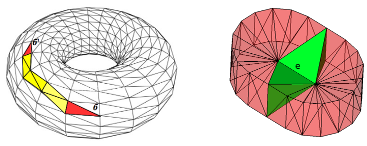
In this section, we briefly discuss some *structural models*, specific for non-manifold shapes.

The *Selective Geometric Complexes* (Rossignac and O'Connor, 1989) exploit decompositions of non-manifold shapes, encoded through the IG data structure (Edelsbrunner, 1987), whose cells can be either open or not connected. In (Desaulniers and Stewart, 1992) the authors propose a decomposition of non-manifold shapes into regular parts, providing interesting topological information. In (Falcidieno and Ratto, 1992) the authors discuss the identification of form features in simplicial shapes, decomposing them into regular parts. In (Gueziec et al., 1998) the authors propose a decomposition-based technique to convert non-manifold shapes into manifolds without addressing geometric aspects. In (Rossignac and Cardoze, 1999) the authors improve over this technique by taking in account also geometry: the key idea is to split and duplicate non-manifold singularities in order to avoid self-intersections. A further extension to volumetric shapes is proposed in (Attene et al., 2009), which applies local updates on the neighborhood of a non-manifold singularity. In (Pesco et al., 2004) the authors propose a non-unique combinatorial stratification of a cell 2-complex on which they define editing operators. The *Initial Quasi-Manifold (IQM-)* decomposition (De Floriani et al., 2003) is a unique and dimension-independent decomposition of abstract simplicial d-complexes. A simplicial shape is decomposed into *IQM-components*, which are regular simplicial d-complexes such that any pair of d-simplices in the star of each vertex is connected by a $(d-1)$-path such that two consecutive d-simplices share a $(d-1)$-simplex. A graph-based data structure for representing the IQM decomposition of any shape embedded in the Euclidean space $\mathbb{E}^3$ is the *Double-Level Decomposition (DLD)* data structure (Hui et al., 2006). In this decomposition, each IQM-component is encoded through the *Indexed data structure with Adjacencies (IA)* (Paoluzzi et al., 1993).

4 THE MC-DECOMPOSITION

In this section, we describe a dimension-independent extension of the *Manifold-Connected (MC-) decomposition*, initially defined in (Hui and De Floriani, 2007) only for 2D and 3D shapes. This decomposition is defined on abstract simplicial complexes.

In a regular simplicial d-complex Σ, two d-simplices σ and σ' are said to be *manifold-connected* if and only if there exists a $(d-1)$-path joining σ and σ' such that two consecutive d-simplices share a manifold $(d-1)$-simplex. A $(d-1)$-simplex in Σ is *manifold* if its star contains at most two d-simplices. We call a $(d-1)$-path with such properties as a *manifold $(d-1)$-path*. Figure 1(a) shows a manifold 1-path (in yellow) between two triangles σ and σ' (in red) in a regular simplicial 2-complex, namely a torus. A regular d-complex Σ, such that any pair of d-simplices is manifold-connected, is a *Manifold-Connected complex (MC-complex) of dimension d*. It is clear that any manifold is also an MC-complex, like torus in Figure 1(a), but the reverse is not true. In fact, an MC-complex may contain non-manifold singularities, like non-manifold edge e in MC-complex of dimension 3 in Figure 1(b). This shows that the class of MC-complexes is a superclass of combinatorial manifolds.

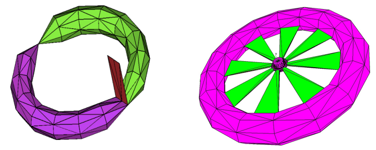


(a) (b)

Figure 1: (a) A manifold 1-path (in yellow) connecting two triangles σ and σ' (in red) in a regular simplicial 2-complex, which is also an MC-complex of dimension 2. (b) An example of MC-complex of dimension 3, which is pinched at a non-manifold edge e. All the tetrahedra incident at e are also highlighted (in green).

The MC-decomposition is defined as the decomposition of any regular simplicial d-complex $\mathcal{X}$ into a collection of MC-complexes of dimension d, called the *MC-components* of $\mathcal{X}$, which can be described in terms of top d-simplices. Manifold-connectivity relation, restricted to top d-simplices in $\mathcal{X}$, defines an equivalence relation on $\mathcal{X}$. MC-components of $\mathcal{X}$ are the equivalence classes of top d-simplices with respect to the manifold-connectivity relation. Specifically, any top d-simplex σ, called the *representative simplex* of $\mathcal{X}$, is equivalent to all the top d-simplices in $\mathcal{X}$, which are reachable from σ through a manifold $(d-1)$-path. The collection of all the MC-components in $\mathcal{X}$ form the *Manifold-Connected (MC-) decomposition* of $\mathcal{X}$. This decomposition is unique, since any top d-simplex σ in $\mathcal{X}$ belongs to only one MC-component. Several MC-components in $\mathcal{X}$ may have a common intersection, formed by non-manifold singularities, i.e., a subcomplex of $\mathcal{X}$ of dimension lower than d. Figure 2(a) shows the MC-decomposition of a regular simplicial 2-complex, which is formed by three MC-components of dimension 2, respectively, in red, purple, and green. These MC-components are connected by two non-manifold vertices and a non-manifold edge.

A non-regular simplicial d-complex Σ is uniquely decomposed into maximal regular subcomplexes Σ_p^t, i.e., collections of top p-simplices in Σ, with $0 < p \leq d$. The *Manifold-Connected (MC-) decomposition* of Σ consists of the MC-decompositions of all the subcomplexes Σ_p^t. Note that each top p-simplex in Σ belongs to only one subcomplex Σ_p^t, thus the MC-decomposition of Σ is unique. Hence, Σ is decomposed into MC-complexes of dimension p, for $0 < p \leq d$, which may have a common intersec-



(a) (b)

Figure 2: (a) Three MC-components of dimension 2 for a regular simplicial 2-complex, respectively, in red, purple, and green. (b) MC-components of dimension 2 (in green) and 3 (in purple) for a non-regular simplicial 3-complex.

tion, formed by non-manifold simplices. Figure 2(b) shows the MC-decomposition of a non-regular simplicial 3-complex, formed by several MC-components of dimension 2 (in green) and 3 (in purple), which are connected by chains of non-manifold edges.

5 THE COMPACT MC-GRAPH

The MC-decomposition for an abstract simplicial d-complex Σ can be represented through a two-level data structure. At the lower level, we use any topological data structure $\mathcal{M}_\Sigma$ for non-manifold shapes to encode Σ. On the contrary, the upper level encodes the connectivity among MC-components through a graph-based data structure. An MC-component $\mathcal{C}$ in Σ corresponds to a node, containing a reference to the representative top simplex of $\mathcal{C}$, while any arc corresponds to the common intersection $\mathcal{S}$ of several MC-components, and contains references to non-manifold simplices in Σ. The resulting representation acts as a sort of spatial index (Samet, 2006), imposed on any topological data structure $\mathcal{M}_\Sigma$. Note that if Σ is manifold, then its MC-decomposition consists of only one MC-component. Hence, only a reference to one top simplex in Σ is stored.

Thus, the main difference among several graph-based representations of the MC-decomposition depends on the encoding of hyperarcs, since nodes are described uniquely. The first graph-based representation of the MC-decomposition, satisfying these design choices, has been proposed in (Boltcheva et al., 2011), but it may become verbose due to the presence of cliques (Canino, 2012).

The *Exploded Manifold-Connected (MC-) graph*, introduced in (Canino and De Floriani, 2011), is a hypergraph $\mathcal{G}^E_\Sigma = (\mathcal{N}_\Sigma, \mathcal{A}^E_\Sigma)$, such that each hyperarc corresponds to a non-manifold simplex σ, and connects n_σ nodes, corresponding to the MC-components incident at σ. We denote the collection of non-manifold singularities in Σ as Σ^n. Let n_E and a_E be, respectively, the number of nodes and arcs in $\mathcal{G}^E_\Sigma$, then the storage cost S_E of the exploded MC-graph, expressed in terms of references, is equal to:

$$S_E = n_E + \sum_{a \in \mathcal{A}^E_\Sigma} (1 + n_\sigma) = n_E + a_E + \sum_{\sigma \in \Sigma^n} n_\sigma \quad (1)$$

Figure 3(b) shows the exploded MC-graph, which represents the MC-decomposition of non-regular 2D shape in Figure 3(a). In this case, there are three hyperarcs, related, respectively to non-manifold vertices v_1 and v_2, and to non-manifold edge e_1. For instance, hyperarc in red relates non-manifold vertex v_2 and MC-components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$, which are incident at v_2. The exploded MC-graph is robust with respect the presence of cliques (Canino, 2012). However, a subset of MC-components may be duplicated and related to several hyperarcs, increasing its storage cost (Canino, 2012). For instance, in Figure 3(b), MC-components $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are connected twice by hyperarcs related to non-manifold vertices v_1 and v_2. Thus, the number of redundancies must be reduced in order to define a compact representation of the MC-decomposition.

Here, we propose the *Compact Manifold-Connected (MC-) graph*, which overcomes all the drawbacks of the exploded MC-graph. The key idea consists of grouping together all the hyperarcs of the exploded MC-graph, which are related to the same subcomplex of non-manifold singularities, in order to reduce the storage cost. The compact MC-graph is a hypergraph $\mathcal{G}^C_\Sigma = (\mathcal{N}_\Sigma, \mathcal{A}^C_\Sigma)$, such that any node corresponds to one MC-component $\mathcal{C}$, while any hyperarc describes the maximal set of non-manifold singularities, shared by MC-components $\mathcal{C}_1, \ldots, \mathcal{C}_k$. Thus, any hyperarc $a = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ satisfies the following properties:

i) intersection $\Sigma^a = \bigcap_{i=1,\ldots,k} \mathcal{C}_i$, is a subcomplex of Σ, not necessarily connected;

ii) there is no MC-component $\mathcal{C}_s \neq \mathcal{C}_i$, for all $i = 1,\ldots,k$, such that $\Sigma^a = \left(\bigcap_{i=1,\ldots,k} \mathcal{C}_i\right) \cap \mathcal{C}_s$.

For each node, corresponding to one MC-component $\mathcal{C}$, one reference to the representative simplex of $\mathcal{C}$ is stored. For any hyperarc a, $k^a = k$ references to MC-components $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are stored, plus s^a references to non-manifold singularities in Σ^a. Let n_C and a_C be, respectively, the number of nodes and hyperarcs in $\mathcal{G}^C_\Sigma$, then the storage cost S_C of the compact MC-graph, expressed in terms of references, is equal to:

$$S_C = n_C + \sum_{a \in \mathcal{A}^C_\Sigma} (k^a + s^a) \quad (2)$$

It is clear that the exploded and the compact MC-graph have the same number of nodes, thus $n_C = n_E$. Figure 3(c) shows the compact MC-Graph, which represents the MC-decomposition of non-regular 2D

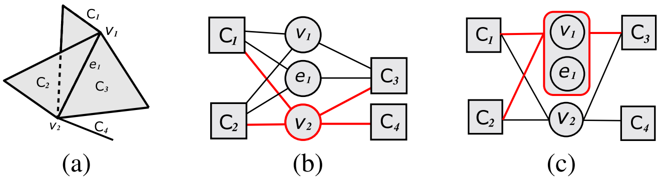


Figure 3: (a) The MC-decomposition of a non-regular 2D shape is formed by four MC-components, connected through non-manifold edge e_1 and vertices v_1 and v_2. Some arcs (in red) in the corresponding (b) exploded and (c) compact MC-graphs.

shape in Figure 3(a). The resulting hypergraph is more compact than the corresponding exploded MC-graph, shown in Figure 3(b). Here, hyperarc $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ (in red) is related to non-manifold singularities v_1 and e_1, while other hyperarc $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4)$ is related only to non-manifold vertex v_2, since $\mathcal{C}_4$ is not incident at v_1 and e_1.

The compact MC-graph, if combined with any topological data structure $\mathcal{M}_\Sigma$, can be computed in two steps. In the first step, all the MC-components are retrieved as described in (Boltcheva et al., 2011). Recall that the identification of MC-components is always defined and completely dimension-independent. We encode the result of the first step as an array L such that, for each non-manifold singularity σ, any location $L[\sigma]$ contains l_σ labels of MC-components incident at σ. Note that this allows defining the exploded MC-graph immediately. In the second step, all the hyperarcs in the compact MC-graph are retrieved by analyzing each location $L[\sigma]$ through an auxiliary array $\mathcal{B}$, as follows:

1. If $L[\sigma]$ contains only one label $\mathcal{C}$, create a tuple $(\sigma, \mathcal{C})$ in $\mathcal{B}$, describing a self-loop related to σ. Otherwise, sort $L[\sigma]$ in increasing order as a list $\bar{l_\sigma}$, and create a tuple $(\sigma, \bar{l_\sigma})$ in $\mathcal{B}$.

2. Sort all the tuples in $\mathcal{B}$ with respect to the lexicographic order of lists of labels. Tuples related to the same subset of labels are stored in consecutive locations of $\mathcal{B}$.

3. Create and complete a new hyperarc for each subset of MC-components, identified at Step 2.

The identification of MC-components depends on the topological data structure $\mathcal{M}_\Sigma$, used in combination with the compact MC-graph. The IA* data structure (Canino et al., 2011) offers the best support for these operations (Canino, 2012). On the contrary, retrieving the hyperarcs does not depend on $\mathcal{M}_\Sigma$. The time complexity of Step 1 is $O(l_\sigma \log l_\sigma)$ for each non-manifold simplex σ, since $L[\sigma]$ is sorted. In Step 2, s^n locations of $\mathcal{B}$ are sorted, one for each non-singularity σ in Σ (i.e., in Σ_n), thus the time complexity of this step is $O(s^n \log s^n)$. Step 3 can performed in $O(1)$. Hence, the time complexity required for computing hyperarcs in the compact MC-graph is linear in:

$$s^n \log s^n + \sum_{\sigma \in \Sigma_n} l_\sigma \log l_\sigma \qquad (3)$$

6 EXPERIMENTAL RESULTS

In our experiments, we have combined the compact MC-graph with all the data structures within the *Mangrove Topological Data Structure (Mangrove TDS) framework* (Canino, 2012). This latter is a framework for the fast prototyping of any topological data structure, representing simplicial complexes without restrictions. A C++ implementation of this framework is contained in the *Mangrove TDS Library* (Canino and De Floriani, 2012), released in public domain. Here, we focus our attention only on the *Incidence Simplicial (IS)* data structure (De Floriani et al., 2010) and the *Generalized Indexed data structure with Adjacencies (IA*)* (Canino et al., 2011), which offer the best performance with respect to queries efficiency and storage cost (Canino, 2012). Note that the IS data structure represents all the simplices explicitly, while the IA* data structure offers a more compact representation by encoding only vertices and top simplices.

In Table 1, we provide experimental comparisons on our graph-based data structures for a subset of simplicial shapes, freely available from (Hui and De Floriani, 2009). First, we compare the storage costs S_E and S_C of the exploded and the compact MC-graph, respectively. As expected, our results in Table 1 show that the compact MC-graph is more compact than the exploded MC-graph: $S_E \approx 2 \times S_C$, on average. This is due to the clustering policy of MC-components, connected by hyperarcs of the compact MC-graph, as discussed in Section 5. Note that the number of hyperarcs a_C in the compact MC-graph is smaller than the number a_E of hyperarcs in the exploded MC-graph. For instance, for the 2D shape *"Tower"* in Table 1, $a_E = 1.4k$ and $a_C = 165$, while $S_E \approx 2.8 \times S_C$.

We also compare the total storage costs S^C_{IS} and $S^C_{IA^*}$ of the compact MC-graph, combined with the IS and the IA* data structures, respectively, and the

Table 1: Statistics on the MC-decomposition and the corresponding exploded and compact MC-graphs of some simplicial 2- and 3-complexes, freely available from (Hui and De Floriani, 2009). Here, column s^t shows the number of top simplices in the input shape, n_c is the number of nodes, while a_E and a_C are, respectively, the number of arcs in the exploded and in the compact MC-graphs. Columns S_E and S_C contain the storage costs of our graph-based representations, expressed as the number of references. In addition, columns S_{IA^*}, S_{IS} and S_{IG} show, respectively, the storage costs of the IA*, IS, and IG data structures. Finally, $S^C_{IA^*}$ and S^C_{IS} denote the storage costs of the compact MC-graph, combined, respectively, with the IA* and the IS data structures. In other words, $S^C_{IA^*} = S_C + S_{IA*}$ and $S^C_{IS} = S_C + S_{IS}$.

Shape	s^t	n_c	a_E	a_C	S_E	S_C	S_{IA^*}	S_{IS}	$S^C_{IA^*}$	S^C_{IS}	S_{IG}
Carter	$8k$	45	641	48	$3.8k$	$1.2k$	$52k$	$75k$	$53.2k$	$76.2k$	$95k$
Chandelier	$8.1k$	130	616	96	$2.6k$	$1k$	$120k$	$174k$	$121k$	$175k$	$220k$
Pinched Pie	$2.3k$	120	$1.4k$	192	$4.8k$	$1.9k$	$17k$	$20.5k$	$18.9k$	$22.4k$	$25k$
Tower	$19k$	169	$1.4k$	165	$5.9k$	$2.1k$	$124k$	$175k$	$126.1k$	$177.1k$	$221k$
Chime	376	27	29	28	133	127	$3.2k$	$8.5k$	$3.3k$	$8.6k$	$12k$
Flasks	$4k$	8	76	6	300	98	$29k$	$75k$	$29.1k$	$75.1k$	$104k$
Teapot	$13k$	$2.9k$	$1.2k$	$1k$	$10.4k$	$10.1k$	$85k$	$163k$	$95.1k$	$173.1k$	$220k$
Wheel	$1.2k$	115	136	88	675	563	$9.9k$	$23.7k$	$10k$	$24k$	$33.4k$

Table 2: Statistics on storage costs of the compact MC-graph, combined with the IA* data structure, and of the IQM-decomposition (De Floriani et al., 2003), for several versions of simplicial shape *"Sierpinski"*, embedded in the Euclidean space $\mathbb{E}^d$, with $2 \leq d \leq 5$. Here, columns s_0 and s_d contain, respectively, the numbers of vertices and of d-simplices in the input shape. Columns S_C, S^*_{IA}, $S^C_{IA^*}$, and S_{IG} are the same as in Table 1. Finally, column S_{IQM} contains the storage cost of the IQM-decomposition.

Shape	s_0	s_d	S_C	S_{IA^*}	$S^C_{IA^*}$	S_{IQM}	S_{IG}
Sierpinski 2D	$88.5k$	$59k$	$324k$	$345k$	$669k$	$757k$	$1.1M$
Sierpinski 3D	$131k$	$65.5k$	$458k$	$524k$	$0.98M$	$1.1M$	$3.67M$
Sierpinski 4D	$195.3k$	$78.1k$	$664k$	$781k$	$1.44M$	$1.64M$	$11.6M$
Sierpinski 5D	$140k$	$46.6k$	$467k$	$559.6k$	$1M$	$1.16M$	$7.7M$

storage cost S_{IG} of the *Incidence Graph (IG)* (Edelsbrunner, 1987). This latter represents all the simplices explicitly, and is commonly used in the applications (Popovic and Hoppe, 1997). Our experimental results in Table 1 (see columns S^C_{IS}, $S^C_{IA^*}$, and S_{IG}) show that the compact MC-graph is more cost effective than the incidence graph. In fact, $S_{IG} \approx 1.12 \times S^C_{IS}$ and $S_{IG} \approx 1.45 \times S^C_{IA^*}$, for 2D shapes, on average, while, for 3D shapes, $S_{IG} \approx 1.3 \times S^C_{IS}$ and $S_{IG} \approx 3.2 \times S^C_{IA^*}$. Columns $S^C_{IA^*}$ and S_{IG} in Table 2 show how the compact MC-graph, combined only with the IA* data structure, continues to be more cost effective than the incidence graph in higher dimensions. For instance, $S_{IG} \approx 14.8 \times S^C_{IA^*}$ for simplicial 4-complexes. This result is very interesting, since the IG data structure, unlike the compact MC-graph, does not explicitly describe the structure of a shape, and does not allow for an efficient identification of non-manifold singularities (Canino, 2012).

Finally, we have also compared our compact MC-graph, combined with the IA* data structure, and the IQM-decomposition (De Floriani et al., 2003) for simplicial d-complexes, embedded in the Euclidean space $\mathbb{E}^d$. For the sake of simplicity, we focus our attention on *Sierpinski* d-shape, embedded in $\mathbb{E}^d$, which is formed by s_0 vertices and s_d d-simplices. Figure 4 shows a simplified 2D version of this shape. Here, non-manifold singularities occur only at vertices, and only $d+1$ vertices are manifold. Any non-manifold vertex v is shared by two d-simplices, and each of them is both a MC- and an IQM-component.

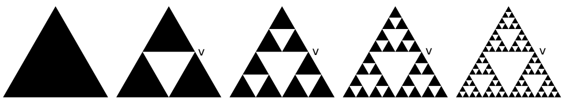


Figure 4: Simplified 2D versions of the *Sierpinski* shape, embedded in the Euclidean space $\mathbb{E}^2$. In this shape, any non-manifold vertex v is shared by two triangles (in black).

The most relevant difference between these two representations consists of the different encoding for the connectivity of subcomponents. Each node of the IQM data structure contains one d-simplex, encoded as an IA data structure (Paoluzzi et al., 1993), thus it requires $(2d+3)s_d$ references. Each arc corresponds to a non-manifold vertex v, and connects two nodes, one for each d-simplex incident at v. In addition, for each arc, two copies of v are stored in order to guarantee that any d-simplex is a valid IQM-component. Hence, the storage cost S_{IQM} of the IQM data structure is $S_{IQM} = (2d+3)s_d + 4(s_0 - d - 1)$. On the contrary, in the compact MC-graph, each simplex is stored only once in the underlying IA* data structure, while, for each arc, references to two MC-components and to a non-manifold vertex are stored. Hence, the storage cost of the compact MC-graph is equal to $S^C_{IA*} = (d+2)s_d + 5s_0 - 4(d+1)$. Experimental results in Table 2 (see columns S^C_{IA*} and S_{IQM}) show that the compact MC-graph tends to be more cost effective than the IQM data structure for any dimension. For instance, $S_{IQM} \approx 1.14 \times S^C_{IA*}$ and $S_{IQM} \approx 1.16 \times S^C_{IA*}$ for, respectively, simplicial 4- and 5-complexes. Note that these representations coincide when encoding manifolds, since they are equivalent to the IA data structure, plus one additional reference to a top simplex in the input shape.

In addition, our experimental results in Table 2 (see columns S_{IQM} and S_{IG}) show that also the IQM data structure is more compact than the incidence graph in any dimension, e.g., $S_{IG} \approx 8 \times S_{IQM}$ for simplicial 4-complexes.

7 CONCLUDING REMARKS

We have presented a structural model for non-manifold shapes, which are decomposed into a collection of MC-components, a decidable superclass of manifolds of any dimension. We have designed and implemented the *Compact MC-graph*, a graph-based representation for the MC-decomposition (Hui and De Floriani, 2007), which can be combined with any topological data structure representing non-manifolds.

We have combined the compact MC-graph with all the topological data structures, which are currently implemented in the *Mangrove TDS Library* (Canino and De Floriani, 2012). Our tests show that the compact MC-graph, if combined with the IS (De Floriani et al., 2010) and the IA* (Canino et al., 2011) data structures, is more compact than the incidence graph (Edelsbrunner, 1987), which is a widely used data structure in several applications. However, this latter, unlike our compact MC-graph, does not expose the structure of a shape explicitly, and does not support the identification of non-manifold singularities efficiently (Canino, 2012). Our tests also show that the compact MC-graph is more cost effective than the IQM-decomposition (De Floriani et al., 2003) and than the DLD data structure (Hui et al., 2006), even for high dimensions.

There is an increasing interest in quad and unstructured hexahedral meshes in geometry processing, animation, and numerical simulations. Some data structures, specific for simplicial complexes, like the IS and IA* data structures, can be easily extended to deal with such shapes, since all the simplifying assumptions, that make the two data structures compact in the case of simplicial complexes, hold also for quad and hexahedral meshes. Thus, also the MC-decomposition can be extended to such meshes and also to more general cell complexes .

Finally, we are designing new graph-based representations for the IQM-decomposition. The properties of the IQM components may allow for a very compact encoding. In fact, an IQM-component is almost manifold, thus it may be representable through very compact data structures, specific for manifolds, like (Gurung and Rossignac, 2009; Gurung et al., 2011a; Gurung et al., 2011b), just to mention few.

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REFERENCES

Attene, M., Giorgi, D., Ferri, M., and Falcidieno, B. (2009). On Converting Sets of Tetrahedra to Combinatorial and PL Manifolds. *Comp.-Aid. Des.*, 26(8):850–864.

Boltcheva, D., Canino, D., Merino Aceituno, S., Léon, J.-C., De Floriani, L., and Hétroy, F. (2011). An Iterative Algorithm for Homology Computation on Simplicial Shapes. *Comp.-Aid. Des.*, 43(11):1457–1467.

Canino, D. (2012). *Tools for Modeling and Analysis of Non-manifold Shapes*. PhD thesis, Department of Computer Science, University of Genova, Genova, Italy.

Canino, D. and De Floriani, L. (2011). A Decomposition-based Approach to Modeling and Understanding Arbitrary Shapes. In *Proc. of the EG Italy*, pages 53–60.

Canino, D. and De Floriani, L. (2012). The Mangrove Topological Data Structure (Mangrove TDS) Library. `http://mangrovetds.sourceforge.net`.

Canino, D., De Floriani, L., and Weiss, K. (2011). IA*: an Adjacency-based Representation for Non-Manifold Simplicial Shapes in Arbitrary Dimensions. *Comp. & Graph.*, 35(3):747–753.

De Floriani, L. and Hui, A. (2005). Data Structures for Simplicial Complexes: an Analysis and a Comparison. In *Proc. of the Symp. on Geom. Proc.*, pages 119–128.

De Floriani, L., Hui, A., Panozzo, D., and Canino, D. (2010). A Dimension-independent Data Structure for Simplicial Complexes. In *Proc. of the 19th Int. Mes. Round.*, pages 403–420.

De Floriani, L., Mesmoudi, M. M., Morando, F., and Puppo, E. (2003). Decomposing Non-manifold Objects in Arbitrary Dimension. *Graph. Mod.*, 65(1/3):2–22.

Desaulniers, H. and Stewart, N. (1992). An Extension of Manifold Boundary Representations to the r-sets. *ACM Trans. on Graph.*, 11(1):40–60.

Edelsbrunner, H. (1987). *Algorithms in Combinatorial Geometry*. Springer.

Falcidieno, B. and Ratto, O. (1992). Two-manifold Cell-decomposition of r-sets. *Comp. Graph. For.*, 11(3):391–404.

Gromov, M. (1987). *Hyperbolic Groups*. Springer.

Gueziec, A., Taubin, G., Lazarus, F., and Horn, W. (1998). Converting Sets of Polygons to Manifold Surfaces by Cutting and Stitching. In *Proc. of the IEEE Conf. on Vis.*, pages 383–390.

Gurung, T., Laney, D., Lindstrom, P., and Rossignac, J. (2011a). SQuad: a Compact Representation for Triangle Meshes. *Comp. Graph. For.*, 30(2):355–364.

Gurung, T., Luffel, M., Lindstrom, P., and Rossignac, J. (2011b). LR: Compact Connectivity Representation for Triangle Meshes. *ACM Trans. on Graph.*, 30(4).

Gurung, T. and Rossignac, J. (2009). SOT: Compact Representation for Tetrahedral Meshes. In *Proc. of the ACM Conf. on Sol. and Phys. Mod.*, pages 79–88.

Hui, A. and De Floriani, L. (2007). A Two-level Topological Decomposition for Non-Manifold Simplicial Shapes. In *Proc. of the ACM Conf. on Sol. and Phys. Mod.*, pages 355–360.

Hui, A. and De Floriani, L. (2009). *The Non-Manifold Meshes Repository*. `http://indy.disi.unige.it/nmcollection`.

Hui, A., Vaczlavik, L., and De Floriani, L. (2006). A Decomposition-based Representation for 3D Simplicial Complexes. In *Proc. of the Symp. on Geom. Proc.*, pages 101–110.

Nabutovsky, A. (1996). Geometry of the Space of Triangulations of a Compact Manifold. *Comm. in Math. Phys.*, 181:303–330.

Paoluzzi, A., Bernardini, F., Cattani, C., and Ferrucci, V. (1993). Dimension-Independent Modeling with Simplicial Complexes. *ACM Trans. on Graph.*, 12(1):56–102.

Pesco, S., Tavares, G., and Lopes, H. (2004). A Stratification Approach for Modeling Two-dimensional Cell Complexes. *Comp. & Graph.*, 28:235–247.

Popovic, J. and Hoppe, H. (1997). Progressive Simplicial Complexes. In *Proc. of the ACM SIGGRAPH*, pages 217–224.

Rossignac, J. and Cardoze, D. (1999). Matchmaker: manifold BReps for Non-manifold R-sets. In *Proc. of the ACM Symp. on Sol. Mod. and Appl.*, pages 31–41. ACM Press.

Rossignac, J. and O'Connor, M. (1989). A Dimension-independent Model for Point-sets with Internal Structures and Incomplete Boundaries. In *Geom. Mod. for Prod. Eng.*, pages 145–180. North-Holland.

Samet, H. (2006). *Foundations of Multidimensional and Metric Data Structures*. Morgan Kaufmann.

Shamir, A. (2008). A Survey on Mesh Segmentation Techniques. *Comp. Graph. For.*, 27(6):1539–1556.

Thakur, A., Banerjee, A. G., and Gupta, S. K. (2009). A Survey of CAD Models Simplification Techniques for Physics-based Simulation Applications. *Comp.-Aid. Des.*, 41(2):65–80.